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Riemannian Geometry of Strong-Laser Plasma

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The optical metric for a strong-laser plasma is derived. The affine connection and curvature related to the optical metric are given and their spatial distributions are studied numerically.

1. INTRODUCTION

As is well known, Einstein's general relativity is an elegant theory for describing the four-dimensional space-time structure. All physical laws and equations obeyed by physical processes are formulated in general covariant form, and the space-time structure is described by the gravitational metric. According to this theory, light propagating through the gravitational field will be curved. Correspondingly, the optical properties of a plasma and other moving media are characterized by their permittivity and permeability. Light traveling through inhomogeneous moving media will be deflected as if the space-time is curved. The similarity of the optical phenomena occurring in a gravitational field and in a moving medium can be described by the optical metric (Gordon, 1923).

The optical metric has been used to study problems such as the general relativistic ponderomotive force in a moving medium (Zhu and Shen, 1987), the electron energy gain in a beat wave laser accelerator (Zhu *et al.*, 1989), the frequency matching in a laser accelerator (Zhu, 1989), the effect of medium background on hydrogen spectra (Zhu and Shen, 1988), the electron spectra in a strong-laser plasma (Zhu, 1992) and the theory of nonlinear medium susceptibility and its possible experimental study (Zhu and Shen,

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1991). It is shown that the introduction of the optical metric not only can explain many physical phenomena in the interaction of a strong field with matter, but generalizes the physical model to curved space-time including a gravitational field.

The behavior of light can be described by using the geodesic equation and geodesic deviation equation. The affine connection and curvature appearing in these equations can be derived as usual if the light–medium system is regarded as a "curved" space-time governed by the optical metric. It is necessary to develop the geometrical properties of the space-time with the optical metric in more detail in order to get information on the behavior of light in a moving medium.

In this paper, the optical metric for the high-power laser-plasma system is derived on the basis of the definition given by Gordon (1923). The affine connection and curvature related to the optical metric are given and their spatial distributions are studied numerically.

2. DERIVATION OF THE OPTICAL METRIC, AFFINE CONNECTION, AND CURVATURE FOR THE STRONG-LASER PLASMA

The optical metric is defined as (Gordon, 1923)

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \left(1 - \frac{1}{\epsilon\mu}\right) u_\mu u_\nu \tag{1}$$

where $g_{\mu\nu}$ is the gravitational metric with the signature $(-,+,+,+)$ and the space-time line element $d\tau^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu$, ϵ is the scalar permittivity, μ is the permeability, and u^μ is the four-dimensional velocity of the moving medium.

If the gravitational effect is negligible, the space-time is flat, i.e.,

$$g_{\mu\nu} = \eta_{\mu\nu} \tag{2}$$

$$\eta_{00} = -1 \tag{3}$$

$$\eta_{11} = \eta_{22} = \eta_{33} = 1 \tag{4}$$

The laser plasma is assumed to move along the x direction. Then we have

$$\mu = 1 \tag{5}$$

$$\epsilon = 1 - N \tag{6}$$

$$u^0 = \left[1 - \left(\frac{C_s V_x}{C}\right)^2\right]^{-1/2} = \left[1 - \left(\frac{C_s N_s}{CN}\right)^2\right]^{-1/2} \tag{7}$$

$$u^1 = \left(\frac{C_s V_x}{C}\right)\left[1 - \left(\frac{C_s V_x}{C}\right)^2\right]^{-1/2} = \left(\frac{C_s N_s}{CN}\right)\left[1 - \left(\frac{C_s N_s}{CN}\right)^2\right]^{-1/2} \tag{8}$$

$$u^2 = u^3 = 0 \tag{9}$$

with $N = n/n_c$, $N_s = n_s/n_c$, and $V_x = v_x/C_s$, where n is the plasma density, $n_c = m\omega^2/4\pi e^2$ is the critical density of the plasma, e and m are the electron charge and mass, respectively, ω is the laser frequency, $v_x = dx/dt$ is the plasma velocity relative to the laboratory frame, $C_s = (ZT_e/M)^{1/2}$ is the sound speed, T_e is the electron temperature, Z is the ion charge number, M is the ion mass, C is the velocity of light, and n_s is the plasma density at the sound speed.

For the quasistatic case, the self-consistent plasma density and laser field have been obtained, and the laser intensity is related to the plasma density by (Shen and Zhu, 1988)

$$|A|^2 = |A_s|^2 - 2(N_s^2/N^2 - 2 \ln N_s + 2 \ln N - 1) \tag{10}$$

where $A = |e|E/m\omega v_e$ is the normalized electric field intensity, E is the electric field intensity, $v_e = (T_e/m)^{1/2}$ is the thermal velocity of the electrons, and A_s is the normalized electric field intensity at the sound speed.

Substitution of equations (2)–(9) into equation (1) gives the optical metric

$$\bar{g}_{00} = -1 + \left(1 - \frac{1}{1 - N}\right)\left[1 - \left(\frac{C_s N_s}{CN}\right)^2\right]^{-1} \tag{11}$$

$$\bar{g}_{01} = \frac{C_s N_s}{C(1 - N)}\left[1 - \left(\frac{C_s N_s}{CN}\right)^2\right]^{-1} \tag{12}$$

$$\bar{g}_{11} = 1 - \left(\frac{N}{1 - N}\right)\left(\frac{C_s N_s}{CN}\right)^2\left[1 - \left(\frac{C_s N_s}{CN}\right)^2\right]^{-1} \tag{13}$$

$$\bar{g}_{22} = \bar{g}_{33} = 1 \tag{14}$$

If $(C_s N_s/CN)^2 << 1$, equations (11)–(13) are reduced to

$$\bar{g}_{00} = -\frac{1}{1 - N} \tag{15}$$

$$\bar{g}_{01} = \frac{C_s N_s}{C(1 - N)} \tag{16}$$

$$\bar{g}_{11} = 1 \tag{17}$$

The affine connection in the curved space-time with the gravitational metric $g_{\mu\nu}$ is given by (Weinberg, 1972)

$$\Gamma^{\alpha}_{\lambda\mu} = \frac{1}{2} g^{\alpha\nu}(g_{\mu\nu,\lambda} + g_{\nu\lambda,\mu} - g_{\lambda\mu,\nu}) \tag{18}$$

where $g^{\mu\nu} g_{\lambda\mu} = \delta^{\nu}_{\lambda}$ and $g_{\mu\nu,\lambda}$ is the ordinary derivative of $g_{\mu\nu}$. Similarly, the affine connection in the space-time with the optical metric $\overline{g}_{\mu\nu}$ can be written as

$$\overline{\Gamma}^{\alpha}_{\lambda\mu} = \frac{1}{2} \overline{g}^{\alpha\nu}(\overline{g}_{\mu\nu,\lambda} + \overline{g}_{\nu\lambda,\mu} - \overline{g}_{\lambda\mu,\nu}) \tag{19}$$

where $\overline{g}^{\mu\nu}\overline{g}_{\lambda\mu} = \delta^{\nu}_{\lambda}$, and $\overline{g}_{\mu\nu,\lambda}$ is the ordinary derivative of $\overline{g}_{\mu\nu}$.

From equations (14)–(17) we obtain the nonzero components of the affine connection $\overline{\Gamma}^{\alpha}_{\lambda\mu}$

$$\overline{\Gamma}^{0}_{00} = \frac{kC_sN_s}{2C(1 - N)^2} \frac{dN}{d\xi} \tag{20}$$

$$\overline{\Gamma}^{0}_{01} = \frac{k}{2(1 - N)} \frac{dN}{d\xi} \tag{21}$$

$$\overline{\Gamma}^{0}_{11} = -\frac{kC_sN_s}{C(1 - N)} \frac{dN}{d\xi} \tag{22}$$

$$\overline{\Gamma}^{1}_{00} = \frac{k}{2(1 - N)^2} \frac{dN}{d\xi} \tag{23}$$

$$\overline{\Gamma}^{1}_{01} = -\frac{kC_sN_s}{2C(1 - N)^2} \frac{dN}{d\xi} \tag{24}$$

$$\overline{\Gamma}^{1}_{11} = \frac{k(C_sN_s)^2}{C^2(1 - N)^2} \frac{dN}{d\xi} \tag{25}$$

with $\xi = kx$.

The curvature tensor in the space-time with the optical metric $\overline{g}_{\mu\nu}$ can be expressed as

$$\overline{R}^{\rho}_{\lambda\mu\nu} = -\overline{\Gamma}^{\rho}_{\lambda\mu,\nu} + \overline{\Gamma}^{\rho}_{\lambda\nu,\mu} - \overline{\Gamma}^{\sigma}_{\lambda\mu}\overline{\Gamma}^{\rho}_{\sigma\nu} + \overline{\Gamma}^{\sigma}_{\lambda\nu}\overline{\Gamma}^{\rho}_{\sigma\mu} \tag{26}$$

Inserting equations (20)–(25) into equation (26), we obtain the nonzero components of the curvature tensor

$$\overline{R}^{0}_{001} = -\frac{3k^2C_sN_s}{4C(1 - N)^3}\left(\frac{dN}{d\xi}\right)^2 - \frac{k^2C_sN_s}{2C(1 - N)^2} \frac{d^2N}{d\xi^2} \tag{27}$$

$$\overline{R}^{0}_{010} = \frac{3k^2C_sN_s}{4C(1 - N)^3}\left(\frac{dN}{d\xi}\right)^2 + \frac{k^2C_sN_s}{2C(1 - N)^2} \frac{d^2N}{d\xi^2} \tag{28}$$

$$\overline{R}^0_{110} = \left[\frac{k^2C_s^2N_s^2}{2C^2(1-N)^3} + \frac{3k^2}{4(1-N)^2}\right]\left(\frac{dN}{d\xi}\right)^2 + \frac{k^2}{2(1-N)}\frac{d^2N}{d\xi^2} \tag{29}$$

$$\overline{R}^0_{101} = -\left[\frac{k^2C_s^2N_s^2}{2C^2(1-N)^3} + \frac{3k^2}{4(1-N)^2}\right]\left(\frac{dN}{d\xi}\right)^2 - \frac{k^2}{2(1-N)}\frac{d^2N}{d\xi^2} \tag{30}$$

$$\overline{R}^1_{001} = -\frac{k^2}{2(1-N)^2}\frac{d^2N}{d\xi^2} - \frac{3k^2}{4(1-N)^3}\left(\frac{dN}{d\xi}\right)^2 \tag{31}$$

$$\overline{R}^1_{010} = \frac{k^2}{2(1-N)^2}\frac{d^2N}{d\xi^2} + \frac{3k^2}{4(1-N)^3}\left(\frac{dN}{d\xi}\right)^2 \tag{32}$$

$$\overline{R}^1_{110} = -\frac{k^2C_sN_s}{2C(1-N)^2}\frac{d^2N}{d\xi^2} - \frac{3k^2C_sN_s}{4C(1-N)^3}\left(\frac{dN}{d\xi}\right)^2 \tag{33}$$

$$\overline{R}^1_{101} = \frac{k^2C_sN_s}{2C(1-N)^2}\frac{d^2N}{d\xi^2} + \frac{3k^2C_sN_s}{4C(1-N)^3}\left(\frac{dN}{d\xi}\right)^2 \tag{34}$$

From equations (27)–(34), we can obtain the nonzero components of the Ricci tensor $\overline{R}_{\mu\nu} \equiv \overline{R}^\lambda_{\mu\lambda\nu}$,

$$\overline{R}_{00} = -\frac{k^2}{2(1-N)^2}\frac{d^2N}{d\xi^2} - \frac{3k^2}{4(1-N)^3}\left(\frac{dN}{d\xi}\right)^2 \tag{35}$$

$$\overline{R}_{01} = \overline{R}_{10} = \frac{k^2C_sN_s}{2C(1-N)^2}\frac{d^2N}{d\xi^2} + \frac{3k^2C_sN_s}{4C(1-N)^3}\left(\frac{dN}{d\xi}\right)^2 \tag{36}$$

$$\overline{R}_{11} = \frac{k^2}{2(1-N)}\frac{d^2N}{d\xi^2} + \left[\frac{k^2C_s^2N_s^2}{2C^2(1-N)^3} + \frac{3k^2}{4(1-N)^2}\right]\left(\frac{dN}{d\xi}\right)^2 \tag{37}$$

Then the scalar curvature in the space-time with the optical metric $\overline{g}$ for a high-power laser-plasma system has the form

$$\begin{aligned}\overline{R} &= \overline{g}^{\mu\nu}\overline{R}_{\nu\mu}\\ &= \left[\frac{k^2}{1-N} + \frac{k^2C_s^2N_s^2}{C^2(1-N)^2}\right]\frac{d^2N}{d\xi^2} + \left[\frac{3k^2}{2(1-N)^2} + \frac{2k^2C_s^2N_s^2}{C^2(1-N)^3}\right]\left(\frac{dN}{d\xi}\right)^2\end{aligned} \tag{38}$$

3. COMPUTATIONAL RESULTS

For the quasistatic case, the self-consistent density profile in the high-power laser-plasma has been obtained (Shen and Zhu, 1988) (Fig. 1). The

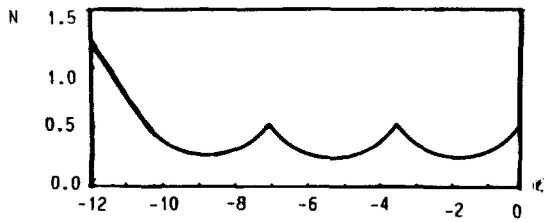

Fig. 1. The self-consistent density in a strong-laser plasma.

spatial distributions of the optical metric, affine connection, and curvature have been investigated numerically, and the computational results are shown in Figs. 2–4.

We see from Fig. 2 that the optical metric components $\bar{g}_{00}$ and $\bar{g}_{01}$ are not continuous at $N = 1$ and tend to infinity. This reflects that the light is stopped at the critical density point. In the region of $N < 1$, $\bar{g}_{00}$ and $\bar{g}_{01}$ exhibit oscillatory behavior with the standing wave structure.

Figure 3 shows the spatial distributions of $\bar{\Gamma}^0_{00}$, $\bar{\Gamma}^0_{01}$, $\bar{\Gamma}^0_{11}$, $\bar{\Gamma}^1_{00}$, $\bar{\Gamma}^1_{01}$, and $\bar{\Gamma}^1_{11}$. As seen, the curves are not continuous at $N = 1$ and $N = N_s$. The curves tend to infinity at $N = 1$ and reach the same numerical values with opposite signs at $N = N_s$ when N comes from the left and right.

Figure 4 shows the spatial distributions of the Ricci curvature tensors ($\bar{R}_{00}$, $\bar{R}_{01}$, and $\bar{R}_{11}$) and the spatial distribution of the scalar curvature $\bar{R}$. The curves are not continuous at $N = 1$, $N = N_s$, and $N = N_{\min}$, and they tend to infinity at $N = 1$ and $N = N_{\min}$.

4. CONCLUSION

The laser plasma is a complex physical object and it is analytically intractable. If the laser-plasma system is regarded as a compound consisting

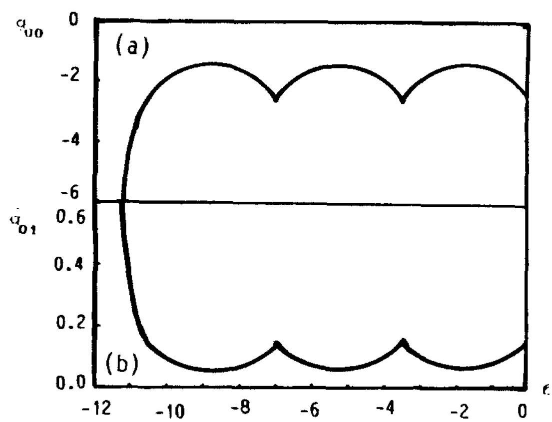

Fig. 2. The spatial distribution of the optical metric (a) $\bar{g}_{00}$ and (b) $\bar{g}_{01}$.

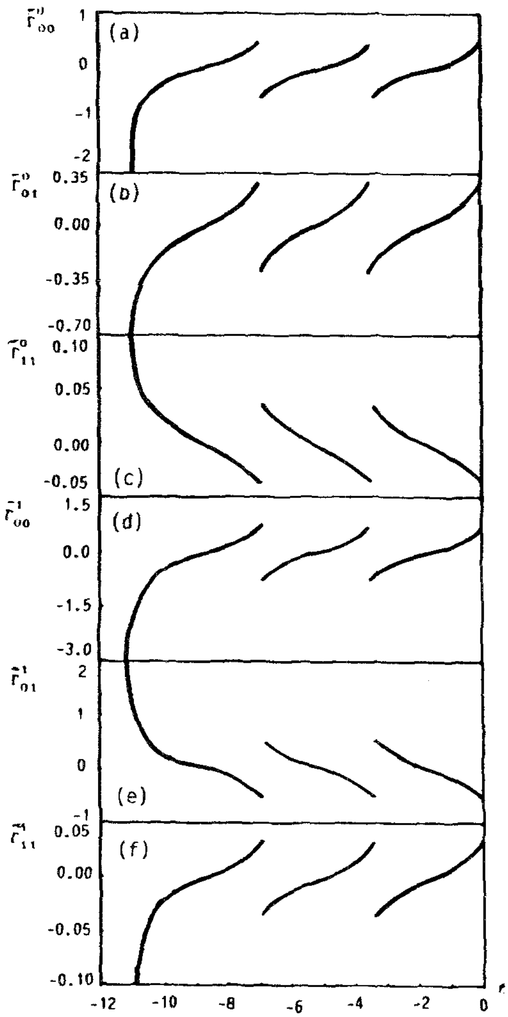

Fig. 3. The spatial distribution of the affine connection [the components (a) $\bar{\Gamma}^0_{00}$, (b) $\bar{\Gamma}^0_{01}$, (c) $\bar{\Gamma}^0_{11}$, (d) $\bar{\Gamma}^1_{00}$, (e) $\bar{\Gamma}^1_{01}$, (f) $\bar{\Gamma}^1_{11}$].

of photons, electrons, and ions embedded in a background characterized by the optical metric, then information on the interaction of light with the plasma helps us to understand the space-time structure of the background with the optical metric and to explain the behavior of the photons, electrons, and ions in such a "curved" space-time.

In this paper, we derived the optical metric, affine connection, and curvature for a strong-laser plasma moving along the x direction, and we studied their spatial distributions numerically. The behavior of the light and

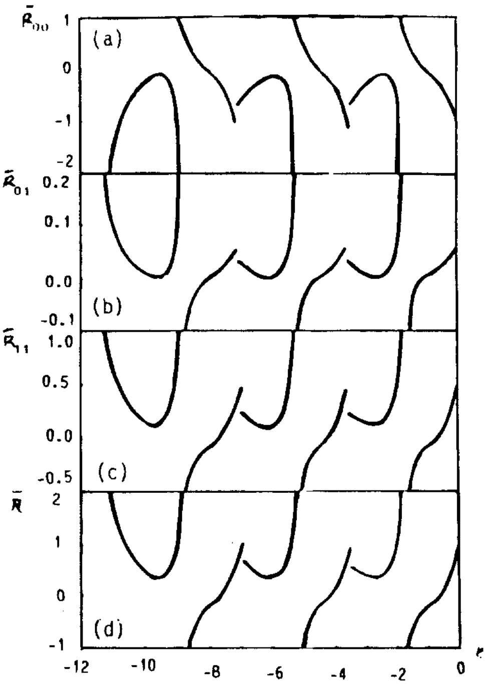

Fig. 4. The spatial distributions of the Ricci curvature tensor [components (a) $\bar{R}_{00}$, (b) $\bar{R}_{01}$, (c) $\bar{R}_{11}$] and (d) the scalar curvature $\bar{R}$.

plasma in the space-time with the optical metric will be discussed elsewhere. Since the optical metric and the gravitational metric are mathematically equivalent, their physical variables and formalism have the same forms; some phenomena in the gravitational field can be simulated by the optical phenomena in a moving medium.

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